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Exact Charged Two-Body Motion and the Static Balance Condition in Lineal Gravity

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Abstract

We find an exact solution to the charged 2-body problem in $(1 + 1)$ dimensional lineal gravity which provides the first example of a relativistic system that generalizes the Majumdar-Papapetrou condition for static balance.

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The static balance problem is a long-standing problem in gravitational physics. Originally motivated by attempts to find exact solutions of the ($N \geq 2$)-body system in general relativity [1], one seeks an equilibrium solution in which gravitational attraction is balanced with another repulsive force, typically electromagnetism. The first such solution was found in Einstein-Maxwell theory by Majumdar [2] and Papapetrou [3] (MJ) for $N = 2$ and was later generalized to N bodies on a line [4]. The MJ static balance condition is

$$e_i = \pm\sqrt{4\pi G}\, m_i \qquad (i = 1, 2) \tag{1}$$

and is considerably more stringent than the corresponding non-relativistic condition

$$Gm_1 m_2 - \frac{e_1 e_2}{4\pi} = 0\ . \tag{2}$$

The reason why these conditions differ have long intrigued theorists. There is no proof that (1) is a necessary condition for static balance, although it is sufficient. Although it has been conjectured [5] that an exact solution under the condition (2) should exist in general relativity, in the (2nd) post-Newtonian approximation (2) is incompatible with the static balance condition [6], and a test particle analysis [7] suggested that (2) is neither necessary nor sufficient. This suggests a wider range of possibilities for realizing equilibrium in general relativity that do not exist non-relativisitically, and several numerical studies [8, 9] have been carried out to this end. Until now no one has yet found – in any relativistic theory of gravity – equilibrium states in which $\sqrt{4\pi G}m_i > e_i$ for both bodies.

We present in this paper a new equilibrium solution to the static balance problem for which the condition (1) does not hold. Our exact solution is obtained for lineal gravity minimally coupled to electromagnetism, and allows for the possibility that the masses of the particles are both larger than their charges. This is the first example of its type, and our full solution is the first non-perturbative relativistic curved-spacetime treatment of this problem, providing new avenues for the study of lineal self-gravitating systems. Indeed, one-dimensional self-gravitating N-body systems have been quite fruitful in yielding insight into many problems in gravity [10]: they admit a considerably simpler level of computational and analytic analysis that can be applied to star systems (small $N \geq 2$) and galactic evolution (large N), whilst avoiding a number of difficulties inherent in three dimensions, including singularities, evaporation, and energy dissipation in the form of gravitational radiation.

Our solution is derived in the context of the canonical theory for a charged N-body relativistic self-gravitating lineal system. We couple N charged point masses to Jackiw-Teitelboim lineal gravity [11], which in the absence of matter equates the scalar curvature to a cosmological constant

$$R - \Lambda = 0\ . \tag{3}$$

As a model theory of quantum gravity [12] this model has been of considerable interest; our modification to include charged particles yields a generally covariant self-gravitating system with non-zero curvature outside the point sources. We do not include collisional terms, so that the bodies pass through each other.

We take the action to be

$$\begin{aligned} I &= \int d^2x \left[\frac{\sqrt{-g}}{2\kappa} g^{\mu\nu} \left\{ \Psi R_{\mu\nu} + \frac{1}{2}\nabla_\mu\Psi\nabla_\nu\Psi + g_{\mu\nu}\Lambda - \frac{\kappa}{2} F_\mu{}^\alpha F_{\nu\alpha} \right\} \right. \\ & \left. - \sum_a \int d\tau_a \left\{ m_a \sqrt{\left(-\frac{dz_{a\mu}}{d\tau_a}\frac{dz_a^\mu}{d\tau_a} \right)} - e_a \frac{dz_a^\mu}{d\tau_a} A_\mu(x) \right\} \delta^{(2)}(x - z_a(\tau_a)) \right] , \end{aligned} \tag{4}$$

where Ψ is the dilaton field, which must be included since the Einstein action is a topological invariant in 2 spacetime dimensions. Here $g_{\mu\nu}$ and g are the metric and its determinant, R is the Ricci scalar, $\kappa = 8\pi G/c^4$, and the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with τ_a the proper time of a-th particle whose mass is m_a and charge is e_a. Variation of the action (4) with respect to the metric, dilaton field, vector potential and particle coordinates yields the field equations

$$R - \Lambda = \kappa T^\mu{}_\mu, \; \frac{dz_a^\alpha}{d\tau_a} \nabla_\alpha \left\{ \frac{dz_a^\nu}{d\tau_a} \right\} = \frac{e_a}{m_a} \frac{dz_a^\alpha}{d\tau_a} F^\nu{}_\alpha(z_a) , \tag{5}$$

$$\partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right) = \sum_a e_a \int d\tau_a \frac{dz_a^\mu}{d\tau_a} \delta^2(x - z_a(\tau_a)) , \tag{6}$$

$$\begin{aligned} \frac{1}{2}\nabla_\mu\Psi\nabla_\nu\Psi - g_{\mu\nu}\left(\frac{1}{4}\nabla^\lambda\Psi\nabla_\lambda\Psi - \nabla^2\Psi \right) - \nabla_\mu\nabla_\nu\Psi & \\ = \kappa T_{\mu\nu} + \frac{\Lambda}{2} g_{\mu\nu} , & \end{aligned} \tag{7}$$

where the stress-energy is due to the electromagnetic field and the point masses

$$\begin{aligned} T_{\mu\nu} &= \sum_{a=1}^N m_a \int \frac{d\tau_a}{\sqrt{-g}} g_{\mu\sigma} g_{\nu\rho} \frac{dz_a^\sigma}{d\tau_a} \frac{dz_a^\rho}{d\tau_a} \delta^{(2)}(x - z_a(\tau_a)) \\ & + \left\{ F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right\} \end{aligned}$$

and is conserved. The set (5,6) is a closed system of $N+2$ equations whose solution yields the single metric and electromagnetic degrees of freedom and the N degrees of freedom of the point masses; it reduces to (3) if all the masses m_a and the charges e_a vanish. Note that the evolution of the charged point-masses governs that of the dilaton field via (7). The left-hand side of (7) is divergenceless (consistent with the conservation of $T_{\mu\nu}$), yielding only one independent equation to determine the single degree of freedom of the dilaton.

Working in the canonical formalism where $\gamma = g_{11}, N_0 = (-g^{00})^{-1/2}, N_1 = g_{10}$, and $A_\mu = (-\varphi, A)$, we have been able to extend the exact solution we previously obtained for neutral bodies [13] to the charged case. After elimination of coordinate and gauge degrees of freedom using standard methods [14], the only independent degrees of freedom are the momenta p_i and spatial coordinates z_i of the particles. In the two particle case these reduce to $r \equiv z_1 - z_2$ and $p_1 = -p_2 = p$, and the Hamiltonian is determined from the equation

$$\tanh(\frac{\kappa\mathcal{J}}{8}|r|) = \frac{\mathcal{J}(B_1+B_2)}{\mathcal{J}^2+B_1B_2} \ , \tag{8}$$

where $\mathcal{J}^2 = \left(\sqrt{H^2+\frac{8\Lambda_e}{\kappa^2}} - 2\epsilon\tilde{p}\right)^2 - \frac{8e_1e_2}{\kappa} - \frac{8\Lambda_e}{\kappa^2}$, $B_{1,2} = H - 2\sqrt{p^2+m_{1,2}^2}$, $\tilde{p}_i = p_i \mathrm{sgn}(z_1-z_2)$, and $r \equiv z_1 - z_2$. The parameter $\epsilon = \pm 1$ is a constant of integration associated with the metric degree of freedom and changes sign under time reversal. We have written $\Lambda_e \equiv \Lambda - \frac{\kappa}{4}(\sum_a e_a)^2$, which is an effective cosmological constant in the spacetime. When $\Lambda_e = 0$,the Hamiltonian H reduces to that of two charged point particles in the non-relativistic limit [15], containing that in ref. [10] when $e_a = 0$. For a given $\Lambda_e \geq -(\kappa H)^2/8$, the equation (8) describes the surface in (r, p, H) space of all allowed phase-space trajectories. A given trajectory in the (r, p) plane is uniquely determined by setting $H = H_0$ in (8), since H is a constant of the motion (a fact easily verified by differentiation of (8) with respect to t).

The explicit solution for the field components – although formally the same as that obtained in ref. [13] – is rather complicated, and will be omitted here. However the equations of motion for $\dot{p}_a$ and $\dot{z}_a$ from (5) have an additional Lorentz-force term which yields qualitatively new features. In the 2-body unequal mass case they can be transformed in terms of a new time coordinate to integral form. These integrals cannot be computed in terms of elementary functions except in the equal mass case where the exact solution is

$$p(\tau) = \frac{\epsilon m}{2}\left(f(\tau) - \frac{1}{f(\tau)}\right) \ , \tag{9}$$

with

$$f(\tau) = \begin{cases} \dfrac{\frac{H}{m}\left(1+\sqrt{\gamma_H}\right)\left\{1-\eta e^{\frac{\epsilon\kappa m}{4}\sqrt{\gamma_m}(\tau-\tau_0)}\right\}}{\gamma_e+\sqrt{\gamma_m}+\left(\sqrt{\gamma_m}-\gamma_e\right)\eta e^{\frac{\epsilon\kappa m}{4}\sqrt{\gamma_m}(\tau-\tau_0)}} & \gamma_m > 0, \\ \dfrac{1+\sqrt{\gamma_H}}{\frac{m}{H}\gamma_e + \sigma\left(m-\sigma\frac{\epsilon\kappa H}{8}(\tau-\tau_0)\right)^{-1}} & \gamma_m = 0, \\ \frac{H}{m}(1+\sqrt{\gamma_H})\left[\gamma_e + \sqrt{-\gamma_m}\dfrac{\sigma + \frac{m^2}{H}\sqrt{-\gamma_m}\tan\left[\frac{\epsilon\kappa m}{8}\sqrt{-\gamma_m}(\tau-\tau_0)\right]}{\frac{m^2}{H}\sqrt{-\gamma_m} - \sigma\tan\left[\frac{\epsilon\kappa m}{8}\sqrt{-\gamma_m}(\tau-\tau_0)\right]}\right]^{-1} & \gamma_m < 0, \end{cases}$$

where $d\tau = d\tau_1 = d\tau_2 = \frac{m}{\sqrt{p^2+m^2}}\frac{\mathcal{J}^2}{C}dt$ is the proper time of each particle and

$$\begin{aligned} \gamma_H &= 1 + \frac{8\Lambda_e}{\kappa^2 H^2} \ , \quad \gamma_e = 1 + \frac{2e_1e_2}{\kappa m^2} \ , \\ \gamma_m &= \gamma_e^2 + \frac{8\Lambda_e}{\kappa^2 m^2} \ , \quad \eta = \frac{\sigma - \frac{m^2}{H}\sqrt{\gamma_m}}{\sigma + \frac{m^2}{H}\sqrt{\gamma_m}} \ , \\ \sigma &= (1+\sqrt{\gamma_H})(\sqrt{p_0^2+m^2} - \epsilon p_0) - \frac{m^2}{H}\gamma_e \ , \\ C &= \mathcal{J}^2 - (H - \frac{2\epsilon\tilde{p}}{\sqrt{\gamma_H}})\left\{B + \frac{\kappa}{16}(\mathcal{J}^2 - B^2)r\right\} \ , \end{aligned}$$

with p_0 the initial momentum at $\tau = \tau_0$. It is then straightforward to obtain an exact expression for r as a function of τ by inserting the expression for $p(\tau)$ into (8) and solving for r as a function of τ.

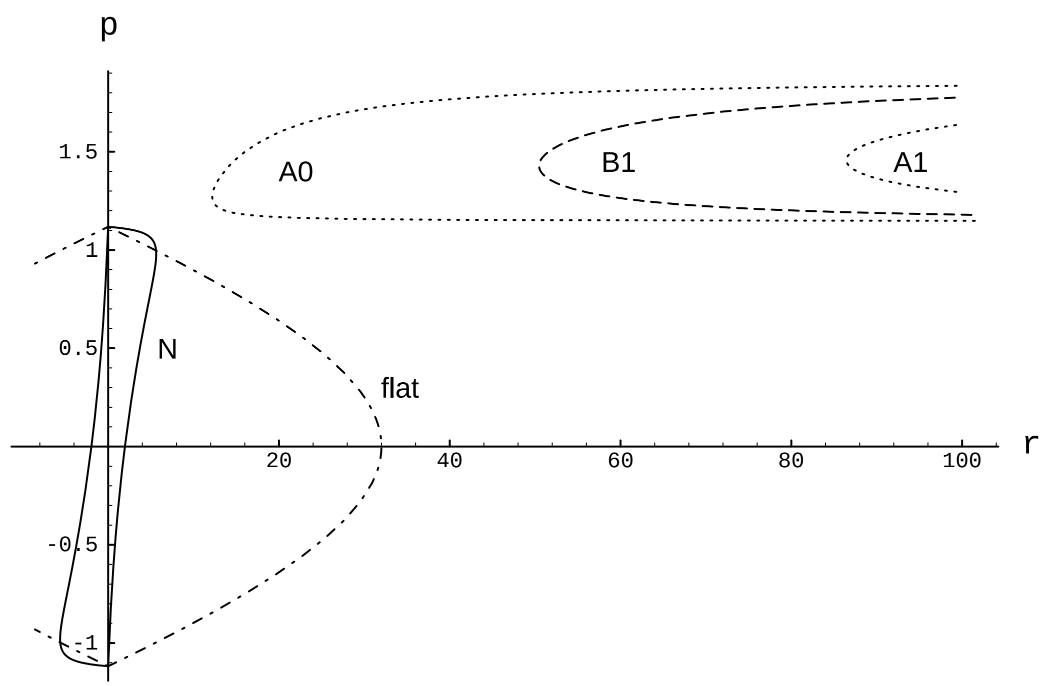

Figure 1: Phase space trajectories of unbounded motions for $H_0 < 2m$ ($m = 1$, $H_0 = 1$ and $e_1 = e_2 = \pm 1$). The $\kappa = 0$ limit is marked *"flat"*.

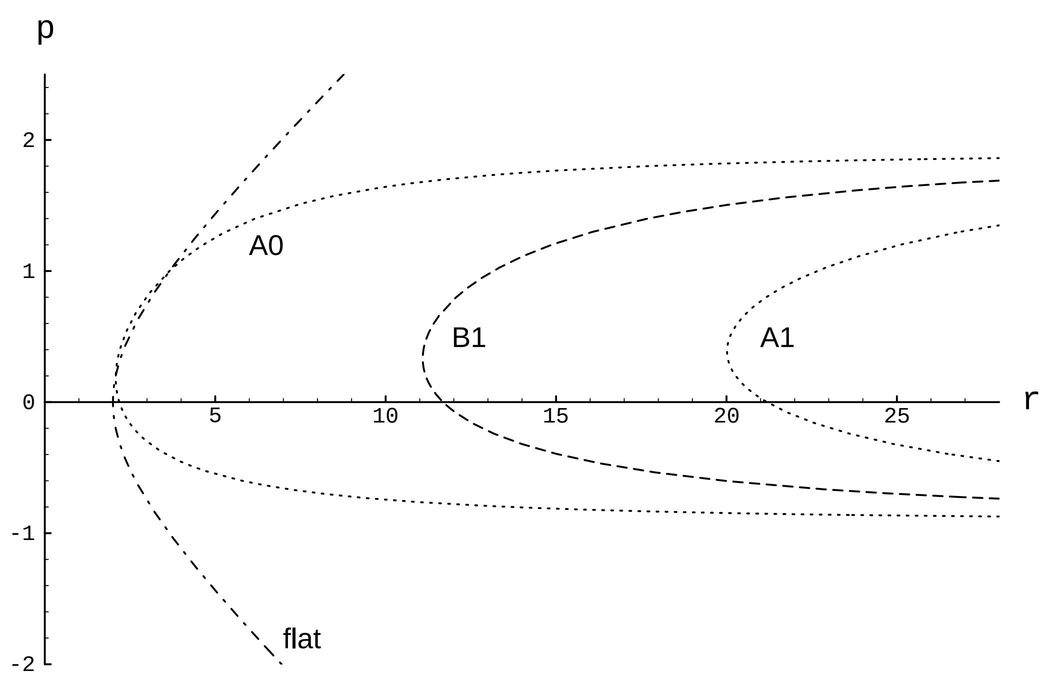

Figure 2: Phase space trajectories of bounded and unbounded motions for $H_0 = 3, m = 1$ and $e_1 = e_2 = \pm 0.25$ in the repulsive case. The $\kappa = 0$ limit is marked *"flat"*.

A number of different types of motion are possible, depending upon a combination of four factors: gravitational attraction, the electric force between charges, the effect of the cosmological constant and relativistic effects. The solution is characterized by the signs of γ_m and $\kappa^2\left(H - mf(\tau) + \frac{m}{f(\tau)}\right)^2 - 8\kappa e_1 e_2 - 8\Lambda_e$.

One of the qualitatively new features which arises is the existence of unbounded motion for $H < 2m$, provided the electromagnetic repulsion between the charges is sufficiently strong. In Fig. 1 we plot (for $\Lambda_e = 0$) several phase space trajectories for $H = m$ and $|e_1e_2| = m(\kappa = 1, \epsilon = 1)$, comparing with the analogous flat-space trajectory (a dot-dashed curve). The additional effect of gravitational attraction causes all trajectories to curve more toward the r-axis and to shift in the direction of the positive p-axis due to the p-and ϵ- dependence of the gravitational potential.

More generally there will only be bounded motion whenever both $e_1e_2 \leq 0$ and $\Lambda_e \leq 0$ in which the electromagnetic and cosmological interactions are attractive, and only unbounded motion whenever these inequalities are reversed. Otherwise both bounded and unbounded motions will be present. Fig.1 is typical for all unbounded motions, and a countably infinite series of unbounded trajectories exists for a fixed value of the total energy H. Fig. 2 illustrates a phase space diagram in the $\Lambda_e = 0$ repulsive case.

Our solution provides the first example of a new equilibrium solution to the static balance problem. From the relation (8) we compute the balance condition $\partial H/\partial r = 0$. This yields the relations

$$\mathcal{J}(B_1 + B_2) = \mathcal{J}^2 + B_1 B_2 = 0 \tag{10}$$

which in turn imply that $B_1 = -B_2$ and $\mathcal{J}^2 = B_1^2$. These simplify to the condition

$$\frac{\kappa}{2}(\sqrt{p^2 + m_1^2} - \epsilon\tilde{p})(\sqrt{p^2 + m_2^2} - \epsilon\tilde{p}) - e_1e_2 = 0 \tag{11}$$

which we refer to as the force-balance condition. Only for $e_1e_2 > 0$ is the value of the momentum fixed

$$p = p_c = \pm \frac{|\left(\frac{\kappa}{2}\right)^2 m_1^2 m_2^2 - e_1^2 e_2^2|}{\sqrt{2\kappa e_1 e_2}\sqrt{(\frac{\kappa}{2}m_1^2 + e_1e_2)(\frac{\kappa}{2}m_2^2 + e_1e_2)}} , \tag{12}$$

in which case the two particles move with constant velocity. Eq. (11) can also be inferred by perturbatively solving for H in terms of r from (8). When the particles are initially at rest ($p_c = 0$), the condition (11) becomes

$$\frac{\kappa}{2}m_1 m_2 - e_1 e_2 = 0 . \tag{13}$$

which is the static balance condition, identical to the non-relativistic condition (2) (also valid in (1+1) dimensions).

The condition (2) applies to both static and uniform motion, whereas the relativistic condition (13) represents only a static balance condition. The condition of force-balance (11) in general depends on the momentum and can be satisfied for some fixed momentum p_c whilst

maintaining for both particles $\sqrt{4\pi G}m_i > e_i$ (or alternatively $\sqrt{4\pi G}m_i < e_i$). This is a qualitatively new feature and suggests that analogous equilibrium states might also exist in (3+1) dimensional general relativity. This is not without precedent: in (3+1) dimensions, for example, a Newtonian theory of gravitating charged particles, if corrected to include the Darwin potential, has an analogous equilibrium solution of constant momentum. It is an interesting open question as to whether or not this feature will survive in a full relativistic theory.

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